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PAPER

A Convolutional Coded ARQ Scheme with Retransmission Criterion Based on an Estimated Decoding Error Rate

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SUMMARY A new Hybrid-ARQ scheme with a convolutional code and the Viterbi decoding is proposed, which uses the packet combining technique and a retransmission criterion based on an estimated decoding error rate. The throughput and bit error rate are evaluated by theoretical bounds and computer simulations. It is shown that a given error rate tolerance can be attained with good throughput for any signal to noise ratio, i.e. for the slow time-varying channels. Furthermore, the throughput performance can be improved by retransmitting not all but a part of packet.

key words: *ARQ, convolutional code, Viterbi decoding, packet combining*

1. Introduction

Error control schemes are roughly classified into the forward error correction (FEC), automatic repeat request (ARQ), and hybrid-ARQ schemes. Among these, the hybrid-ARQ scheme can attain especially high reliability and throughput because it is constructed by combining the advantages of the other two schemes. Various hybrid-ARQ schemes with convolutional code using the Viterbi algorithm or sequential decoding have been proposed and analyzed by many researchers [1]–[12]. In these schemes, retransmission is requested based on the difference of log-likelihoods between the best two paths, the changing rate of log-likelihood of the best path, decoding time out in the sequential decoding, etc. But, since these retransmission criteria are not based on the decoding error rate directly, the error rate becomes worse as the signal to noise ratio (SNR) of a channel degrades. Hence if the SNR is time-varying, it is difficult to satisfy a given error rate tolerance constantly by such schemes though it is not impossible. For instance, we may be able to attain a given error rate tolerance for time-varying channels by preparing multiple convolutional codes or thresholds and changing them based on an estimated SNR. But such configuration maybe introduces high complexity and/or low throughput.

Recently, the packet combining method is proposed as a new technique for ARQ schemes [4]–[6], [13]. In the packet combining, any transmitted packets are never discarded and all retransmitted packets are combined to one packet to improve the performance. In this paper, we propose and analyze a new simple hybrid-ARQ scheme, which uses the packet combining and a retransmission criterion based on an estimated decoding error rate. We show theoretically and by computer simulations that the proposed simple scheme can attain a given error rate tolerance for any SNR with good throughput. Furthermore, the derived theoretical bounds of error rate and throughput are shown to be tight.

If we retransmit a whole packet when the estimated error rate exceeds the tolerance slightly, then the combined packet caused by the retransmission attains much better error rate than we require. In such case, the required error rate can be achieved by retransmitting a part of packet instead of the whole packet. Therefore, we also propose a partial retransmission scheme (PR scheme), in which the number of retransmitted bits varies according to the estimated error rate. This PR scheme considerably improve the throughput compared with the whole retransmission scheme (WR scheme).

In this paper, we assume that a channel is a memoryless additive white Gaussian noise channel which has mean zero and variance $\mathcal{N}_0/2$, and a feedback channel to request retransmission is error free. Furthermore, we assume that the channel is a slow time-varying channel, the SNR of which varies but can be considered constant during a series of retransmissions. However the proposed scheme can also be applied to a moderately time-varying channel where the SNR is constant during one packet transmission but it varies every retransmission. The analysis for such moderately time-varying channel is treated in Ref. [14].

In Sect. 2, the proposed system is described precisely. Some theoretical bounds of the bit error rate and throughput for the WR and PR schemes are derived in Sects. 3 and 4, respectively. Finally, computer simulation results are shown to confirm the validity of the theoretical analyses in Sect. 5.

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2. ARQ Scheme Based on an Estimated Decoding Error Rate

Let $X = (x_1, x_2, \cdots, x_N)$ be a convolutionally coded packet with length N, and let each x_j be transmitted by the BPSK (binary phase shift keying) with transmission power $\mathcal{E}$, i.e. $x_j = \pm\sqrt{\mathcal{E}}$. When the same packet X is transmitted I-times, we represent the i-th received packet by $Y^i = (y_1^i, y_2^i, \cdots, y_N^i)$, $i = 1, 2, \cdots, I$. Note that $I = 1$ means no retransmission. In the packet combining, these received I packets are combined as follows. Each j-th value of these packets is averaged by

$$\tilde{y}_j^I = (y_j^1 + y_j^2 + \cdots + y_j^I)/I, \tag{1}$$

and a new combined packet $\tilde{Y}^I = (\tilde{y}_1^I, \tilde{y}_2^I, \cdots, \tilde{y}_N^I)$ is generated. By applying the Viterbi algorithm to this combined packet $\tilde{Y}^I$, we can obtain an estimated transmitted packet $\tilde{X}^I = (\tilde{x}_1^I, \tilde{x}_2^I, \cdots, \tilde{x}_N^I)$. It is known that this $\tilde{X}^I$ is the maximum likelihood estimation of $\tilde{Y}^I$ [5], [6], [13]. Considering an equivalent channel such that $\tilde{Y}^I$ is received when $\tilde{X}^I$ is transmitted, we can estimate the SNR of the equivalent channel by

$$\overline{\eta}_I = \frac{\mathcal{E}^2}{2\sum_{j=1}^{N} \frac{(\tilde{x}_j^I \tilde{y}_j^I - \mathcal{E})^2}{N}}. \tag{2}$$

In the WR scheme, if this estimated SNR $\overline{\eta}_I$ is greater than a given threshold T, then $\tilde{X}^I$ is accepted as the decoded packet. Otherwise, the retransmission of X is requested. Note that the decoding error rate can be estimated from $\overline{\eta}_I$ because it is determined by the SNR of the channel.

In the PR scheme, a packet with length N is divided into U subblocks, each of which contains $K(= N/U)$ bits. See Fig. 1, where $K = 4$. The retransmission region of $\overline{\eta}_I$, i.e. $\overline{\eta}_I \leqq$ T, is also divided into K regions, which are shown as $R_l \triangleq \{\overline{\eta}_I : \mathrm{T}_{l-1} \leqq \overline{\eta}_I < \mathrm{T}_l\}, l = 1, 2, \cdots, K$, in Fig. 2 where R_A is the acceptance region. If the estimated SNR $\overline{\eta}_I$ is in R_l, then l bits of each block are retransmitted. For example, bit position 'a' shown in Fig. 1 is retransmitted if $\overline{\eta}_I$ is in R_1, or two bit positions 'a' and 'b' are retransmitted if $\overline{\eta}_I$ is in R_2, and so on. In the case that the retransmission is requested more than once, we adjust the bit positions of retransmission in such a way that every position is as equally retransmitted as possible. For instance, if we have already retransmitted positions 'a', 'b', and 'c', then we use position 'd' if $\overline{\eta}_I \in R_1$ or we use positions 'd' and 'a' if $\overline{\eta}_I \in R_2$. In other words, we shift the positions like a,b,c,d,a,b,c,d,$\cdots$.

In the PR scheme, the received packets are combined by

$$\tilde{y}_j^I = \left(\sum_{i=1}^{I} \delta_j^i y_j^i\right) / \sum_{i=1}^{I} \delta_j^i \tag{3}$$

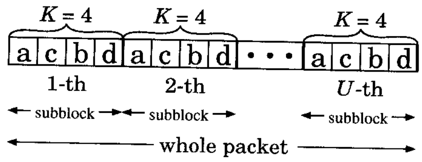

Fig. 1 Example of packet partition.

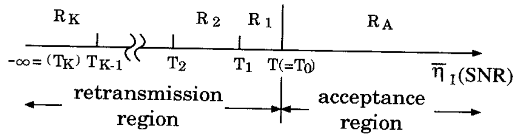

Fig. 2 Partition of retransmission region.

in stead of Eq. (1). Here δ_j^i is defined as

$$\delta_j^i = \begin{cases} 1, & \text{if the } i\text{-th packet contains the } j\text{-th bit of } X, \\ 0, & \text{otherwise.} \end{cases}$$

3. Performance Evaluation of WR Scheme

3.1 Throughput

Let M be the average number of transmitted packets necessary to be finally accepted. Then, the throughput τ of the WR scheme is given by $\tau = R/M$ where R is the rate of convolutional code. Hence, we evaluate M in this subsection. We first note from [4, Eq. (22)] that M can be described as follows.

$$\begin{aligned} M &= 1 \cdot P(S_1) + 2 \cdot P(F_1 S_2) + 3 \cdot P(F_1 F_2 S_3) + \cdots \\ &= 1 \cdot \{P(S_1) + P(F_1) - P(F_1)\} \\ &\quad + 2 \cdot \{P(F_1 S_2) + P(F_1 F_2) - P(F_1 F_2)\} \\ &\quad + 3 \cdot \{P(F_1 F_2 S_3) + P(F_1 F_2 F_3) \\ &\qquad - P(F_1 F_2 F_3)\} + \cdots \\ &= 1 + P(F_1) + P(F_1 F_2) + P(F_1 F_2 F_3) + \cdots, \end{aligned} \tag{4}$$

where events F_I and S_I are defined as

F_I : Event that retransmission is requested for $\tilde{Y}^I$,

S_I : Event that the decoded packet of $\tilde{Y}^I$, i.e. $\tilde{X}^I$, is accepted,

and the last equality holds from $P(F_1 \cdots F_{I-1}) = P(F_1 \cdots F_{I-1} S_I) + P(F_1 \cdots F_{I-1} F_I)$†. Since inequality $P(F_I | F_1 \cdots F_{I-1}) \geqq P(F_I)$ holds for any I [4], [5]††, we have inequality $P(F_1 \cdots F_I) \geqq P(F_1)P(F_2) \cdots P(F_I)$. Therefore, Eq. (4) is bounded below as

†$P(F_1 F_2 \cdots F_I)$ stands for the probability of the event that $F_1, F_2, \cdots, F_I$ jointly occur.

††This inequality can be understood intuitively because conditions $F_1, \cdots, F_{I-1}$ mean that $\tilde{Y}^1, \cdots, \tilde{Y}^{I-1}$ are noisy. Hence $\tilde{Y}^I$ is more noisy when $F_1, \cdots, F_{I-1}$ are conditioned rather than unconditioned.

$$M \geqq 1+\sum_{I=1}^{\infty}\prod_{m=1}^{I} P(F_m). \quad (5)$$

On the other hand, from an obvious inequality $P(F_1F_2\cdots F_I) \leqq P(F_I)$, Eq. (4) is bounded above as

$$M \leqq 1+\sum_{I=1}^{\infty} P(F_I). \quad (6)$$

The upper and lower bounds of $P(F_I)$ will be evaluated in Sect. 3.3.

3.2 Average Bit Error Rate

Let E_I be the event that errors occur when $\tilde{Y}^I$ is decoded by the ordinary Viterbi decoder. Then the event error probability P_E is obtained as follows.

$$\begin{aligned}P_E &= P(S_1E_1)+P(F_1S_2E_2)+P(F_1F_2S_3E_3)+\cdots\\ &= \{P(E_1)-P(F_1E_1)\}\\ &\quad+\{P(F_1E_2)-P(F_1F_2E_2)\}\\ &\quad+\{P(F_1F_2E_3)-P(F_1F_2F_3E_3)\}+\cdots\\ &\leqq \{P(E_1)-P(F_1)P(E_1|F_1)\}\\ &\quad+\{P(E_2)-P(F_1F_2)P(E_2|F_1F_2)\}\\ &\quad+\{P(E_3)-P(F_1F_2F_3)P(E_3|F_1F_2F_3)\}+\cdots\\ &\leqq \sum_{I=1}^{\infty} P(E_I)\left(1-\prod_{m=1}^{I} P(F_m)\right),\end{aligned} \quad (7)$$

where the first inequality follows from $P(F_1F_2\cdots F_{I-1}E_I) \leqq P(E_I)$, and the second inequality holds because of inequalities $P(E_I|F_1F_2\cdots F_I) \geqq P(E_I)$ and $P(F_1F_2\cdots F_I) \geqq P(F_1)P(F_2)\cdots P(F_I)$.

In the same way, the bit error rate P_B is bounded by

$$P_B \leqq \sum_{I=1}^{\infty} P_B^I\left(1-\prod_{m=1}^{I} P(F_m)\right),$$

where P_B^I is the bit error rate of $\tilde{Y}^I$ for the ordinary Viterbi decoding. It is worth noting that in convolutional coded ARQ schemes, some additional bits must be transmitted at the end of packet to merge all surviving paths into one path. Hence a long packet should be used to attain good throughput. In such case, P_B is a better measure of the performance than P_E.

3.3 Retransmission Probability

Retransmission is requested when $\overline{\eta}_I$ calculated by Eq. (2) is less than a given threshold T. Hence the retransmission probability $P(F_I)$ can be described as

$$P(F_I) = \sum_{\tilde{X}^I} P(\tilde{X}^I)P(\overline{\eta}_I < \mathrm{T}|\tilde{X}^I). \quad (8)$$

Since the channel is assumed to be a memoryless additive white Gaussian noise channel, the j-th transmitted symbol x_j and the j-th received symbol y_j^i of the i-th packet are related by $y_j^i = x_j + n_j^i$ where n_j^i stands for the Gaussian random variable with mean zero and variance $\mathcal{N}_0/2$. Note that $\mathcal{N}_0$ is the one-sided noise power spectral density and the SNR η of the channel is given by $\eta = \mathcal{E}/\mathcal{N}_0$. From Eq. (1), the j-th symbol $\tilde{y}_j^I$ of the combined packet $\tilde{Y}^I$ can be represented by

$$\begin{aligned}\tilde{y}_j^I &= x_j + \left(\sum_{i=1}^{I} n_j^i\right)/I\\ &\triangleq x_j + \tilde{n}_j^I\end{aligned} \quad (9)$$

where $\tilde{n}_j^I$ is the Gaussian random variable with mean zero and variance $\mathcal{N}_0/2I$. By substituting Eq. (9) into Eq. (2), we have

$$\overline{\eta}_I = \frac{\mathcal{E}N}{2\sum_{c=1}^{N-d(X,\tilde{X}^I)}\left(\tilde{n}_{j_c}^I\right)^2 + 2\sum_{e=1}^{d(X,\tilde{X}^I)}\left(\tilde{n}_{j_e}^I - 2\sqrt{\mathcal{E}}\right)^2}, \quad (10)$$

where $d(X,\tilde{X}^I)$ denotes the Hamming distance between X and $\tilde{X}^I$, and j_c and j_e represent bit positions j such that $x_j = \tilde{x}_j$ and $x_j \neq \tilde{x}_j$, respectively. From Eqs. (8) and (10), $P(F_I)$ becomes

$$\begin{aligned}P(F_I) &= \sum_{\tilde{X}^I} P(\tilde{X}^I)P\left(\frac{1}{\overline{\eta}_I} > \frac{1}{\mathrm{T}}\middle|\tilde{X}^I\right)\\ &= \sum_{\tilde{X}^I} P(\tilde{X}^I)P\Bigg(\sum_{c=1}^{N-d(X,\tilde{X}^I)}\left(\tilde{n}_{j_c}^I\right)^2\\ &\quad+\sum_{e=1}^{d(X,\tilde{X}^I)}\left(\tilde{n}_{j_e}^I - 2\sqrt{\mathcal{E}}\right)^2 > \frac{N\mathcal{E}}{2\mathrm{T}}\Bigg|\tilde{X}^I\Bigg).\end{aligned} \quad (11)$$

From Lemma A1 in Appendix A, Eq. (11) can be bounded below as follows.

$$\begin{aligned}P(F_I) &> P\left(\overline{\eta}_I < \mathrm{T}|\tilde{X}^I = X\right)\\ &\approx P\left(\sum_{j=1}^{N}\left(\tilde{n}_j^I\right)^2 > \frac{N\mathcal{E}}{2\mathrm{T}}\right).\end{aligned} \quad (12)$$

Since the summation $\sum_{j=1}^{N}\left(\tilde{n}_j^I\right)^2$ is χ^2 distribution with N degrees of freedom, its probability density function is given by

$$p_{\chi^2}(t) = \frac{1}{\left(\frac{\mathcal{N}_0}{2I}\right)^{\frac{N}{2}} 2^{\frac{N}{2}}\Gamma\left(\frac{N}{2}\right)} t^{\frac{N}{2}-1}\exp\left(-\frac{t}{2\frac{\mathcal{N}_0}{2I}}\right). \quad (13)$$

Hence Eq. (12) becomes

$$\begin{aligned} P(F_I) &> \int_{\frac{N\mathcal{E}}{2\mathrm{T}}}^{\infty} \frac{1}{\left(\frac{N_0}{2I}\right)^{\frac{N}{2}} 2^{\frac{N}{2}} \Gamma\left(\frac{N}{2}\right)} t^{\frac{N}{2}-1} \exp\left(-\frac{t}{2\frac{N_0}{2I}}\right) dt \\ &= 1 - \frac{\gamma\left(\frac{N}{2}, \frac{IN}{2\mathrm{T}}\eta\right)}{\Gamma\left(\frac{N}{2}\right)}, \end{aligned} \tag{14}$$

where $\Gamma(\cdot)$ and $\gamma(\cdot)$ are the gamma function and the incomplete gamma function of the first kind, respectively. Since the packet length N is usually more than one hundred bits, Eq. (14) cannot easily be calculated numerically. But, in such case, $p_{\chi^2}(t)$ can be approximated very closely by the Gaussian distribution as follows.

$$p_{\chi^2}(t) \approx \frac{1}{\sqrt{2\pi \cdot 2N(\frac{N_0}{2I})^2}} \exp\left\{-\frac{(t - N\frac{N_0}{2I})^2}{2 \cdot 2N(\frac{N_0}{2I})^2}\right\}. \tag{15}$$

Therefore, $P(F_I)$ has the following bound

$$\begin{aligned} P(F_I) &> P\left(\overline{\eta}_I < \mathrm{T} | \tilde{X}^I = X\right) \\ &\approx \int_{\frac{N\mathcal{E}}{2\mathrm{T}}}^{\infty} \frac{1}{\sqrt{2\pi \cdot 2N(\frac{N_0}{2I})^2}} \exp\left\{-\frac{(t - N\frac{N_0}{2I})^2}{2 \cdot 2N(\frac{N_0}{2I})^2}\right\} dt \\ &= Q\left\{\sqrt{\frac{N}{2}}\left(\frac{I}{\mathrm{T}}\eta - 1\right)\right\} \end{aligned} \tag{16}$$

where $Q(a) = \int_a^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2}\right\} dt$.

Next, we derive an upper bound of $P(F_I)$. From Eqs. (8) and (16) and $P(E_I) = \sum_{\tilde{X}^I \neq X} P(\tilde{X}^I)$, we have

$$\begin{aligned} P(F_I) &= \sum_{\tilde{X}^I \neq X} P(\tilde{X}^I) P(\overline{\eta}_I < \mathrm{T} | \tilde{X}^I) \\ &\quad + \left(1 - \sum_{\tilde{X}^I \neq X} P(\tilde{X}^I)\right) P(\overline{\eta}_I < \mathrm{T} | \tilde{X}^I = X) \\ &< P(E_I) + \{1 - P(E_I)\} P(\overline{\eta}_I < \mathrm{T} | \tilde{X}^I = X) \\ &\approx P(E_I)\left[1 - Q\left\{\sqrt{\frac{N}{2}}\left(\frac{I}{\mathrm{T}}\eta - 1\right)\right\}\right] \\ &\quad + Q\left\{\sqrt{\frac{N}{2}}\left(\frac{I}{\mathrm{T}}\eta - 1\right)\right\}. \end{aligned} \tag{17}$$

3.4 Upper Bounds of P_B^I and $P(E_I)$

In order to represent the well-known upper bounds of P_B^I and $P(E_I)$, we use the following notation.

$\hat{X}_L$: Incorrect path which diverges from the correct path X and remerges exactly after L branches.

$A(X, \hat{X}_L)$: The number of erroneous information bits corresponding to an incorrect path $\hat{X}_L$.

$d = d(X, \hat{X}_L)$: Hamming distance between paths X and $\hat{X}_L$.

$m(\tilde{Y}^I, X) = \sum_{i=1}^{N} \tilde{y}_i^I x_i$: Path metric which is used in the Viterbi decoder.

P_E^I : Event error probability of $\tilde{Y}^I$ for ordinary Viterbi decoding.

Then, it is well known that P_B^I, P_E^I and $P(E_I)$ are bounded as follows.

$$\begin{aligned} P_B^I &< \sum_{L=1}^{\infty} \sum_{\hat{X}_L} A(X, \hat{X}_L) \cdot P(m(\tilde{Y}^I, X) - m(\tilde{Y}^I, \hat{X}_L) < 0) \\ &= \sum_{L=1}^{\infty} \sum_{\hat{X}_L} A(X, \hat{X}_L) P\left(\sum_{l=1}^{L} (x_{jl} - \hat{x}_{jl}) \tilde{y}_{jl}^I < 0\right) \\ &= \sum_{L=1}^{\infty} \sum_{\hat{X}_L} A(X, \hat{X}_L) \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi 4\mathcal{E}\frac{N_0}{2I} d}} \cdot \exp\left\{-\frac{(t - 2\mathcal{E}d)^2}{2 \cdot 4\mathcal{E}\frac{N_0}{2I} d}\right\} dt \\ &= \sum_{L=1}^{\infty} \sum_{\hat{X}_L} A(X, \hat{X}_L) Q\left(\sqrt{2dI\eta}\right), \end{aligned} \tag{18}$$

$$\begin{aligned} P(E_I) &\leq 1 - (1 - P_E^I)^{N-\kappa} \\ &\leq 1 - \left(1 - \sum_{L=1}^{\infty} \sum_{\hat{X}_L} Q\left(\sqrt{2dI\eta}\right)\right)^{N-\kappa}, \end{aligned} \tag{19}$$

where κ denotes the constraint length of the convolutional code.

4. Performance Evaluation of PR Scheme

In this section, we derive the throughput and average bit error rate of the PR scheme.

4.1 Throughput

Since the length of each retransmitted packet is not constant, M is no longer related to the throughput. Hence, instead of M, we consider the normalized average number of transmitted packets, $\overline{M}$, which is defined as $\overline{M} \triangleq$(Expected number of all transmitted bits)/N, because the throughput τ is obtained by $\tau = R/\overline{M}$. In order to evaluate $\overline{M}$, we use the following notation.

q_k^m : k-th transmitted packet caused by the event that $\overline{\eta}_{k-1}$ of $\tilde{Y}^{k-1}$ is in retransmission region R_m. We represent the first transmitted packet by q_1.

$\tilde{Y}^{I(ij\cdots k)}$: Combined packet which is constructed from the received packets corresponding to $q_1 q_2^i q_3^j \cdots q_I^k$.

$F^l_{I(ij\cdots k)}$: Event that $\overline{\eta}_I$ of $\tilde{Y}^{I(ij\cdots k)}$ is in retransmission region R_l, i.e. $\mathrm{T}_l \leqq \overline{\eta}_I < \mathrm{T}_{l-1}$. F_1^l represents the same event for the first transmission.

$S_{I(ij\cdots k)}$: Event that $\overline{\eta}_I$ of $\tilde{Y}^{I(ij\cdots k)}$ is in acceptance region R_A, i.e. $\overline{\eta}_I \geqq \mathrm{T}_0$. S_1 represents the same event for the first transmission.

$a_l = N_l/N$: Packet length ratio normalized by the whole packet length N. N_l stands for the packet length corresponding to retransmission region R_l.

Then, in the similar way as Eq. (4), $\overline{M}$ can be derived as follows.

$$\begin{aligned}\overline{M} &= 1\cdot P(S_1) + \sum_{i=1}^{K}(1+a_i)P(F_1^i S_{2(i)}) \\ &\quad + \sum_{i=1}^{K}\sum_{j=1}^{K}(1+a_i+a_j)P(F_1^i F_{2(i)}^j S_{3(ij)}) + \cdots \\ &= 1 - P(S_1^c) + \sum_{i=1}^{K}(1+a_i)\{P(F_1^i) - P(F_1^i S_{2(i)}^c)\} \\ &\quad + \sum_{i=1}^{K}\sum_{j=1}^{K}(1+a_i+a_j) \\ &\quad \cdot\{P(F_1^i F_{2(i)}^j) - P(F_1^i F_{2(i)}^j S_{3(ij)}^c)\} + \cdots \\ &= 1 + \sum_{i=1}^{K} a_i P(F_1^i) + \sum_{i=1}^{K}\sum_{j=1}^{K} a_j P(F_1^i F_{2(i)}^j) \\ &\quad + \sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{k=1}^{K} a_k P(F_1^i F_{2(i)}^j F_{3(ij)}^k) + \cdots \end{aligned} \tag{20}$$

where S^c represents the complement of event S and the last equality follows from that $P(S_1^c) = \sum_{i=1}^{K} P(F_1^i)$ and so on. We note that the second term of Eq. (20) can be represented by

$$\begin{aligned}\sum_{i=1}^{K} a_i P(F_1^i) &= \sum_{i=1}^{K} a_i P(\mathrm{T}_i \leqq \overline{\eta}_1 < \mathrm{T}_{i-1}) \\ &= \sum_{i=1}^{K}(a_i - a_{i-1})P(\overline{\eta}_1 < \mathrm{T}_{i-1}) \\ &\triangleq \sum_{i=1}^{K}(a_i - a_{i-1})P(\overline{F}_1^i)\end{aligned}$$

where $a_0 = 0$ and $\overline{F}_1^i$ is the event such that $\overline{\eta}_1 < \mathrm{T}_{i-1}$, and hence $P(\overline{F}_1^i)$ is given by

$$P(\overline{F}_1^i) = \sum_{l=i}^{K} P(F_1^l). \tag{21}$$

Since the similar relations hold for the other terms in Eq. (20), we have

$$\begin{aligned}\overline{M} &= 1 + \sum_{i=1}^{K}(a_i - a_{i-1})P(\overline{F}_1^i) \\ &\quad + \sum_{i=1}^{K}\sum_{j=1}^{K}(a_j - a_{j-1})P(F_1^i \overline{F}_{2(i)}^j) \\ &\quad + \sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{k=1}^{K}(a_k - a_{k-1})P(F_1^i F_{2(i)}^j \overline{F}_{3(ij)}^k) + \cdots .\end{aligned} \tag{22}$$

From Eq. (22) and $P(F_1^K F_{2(K)}^K \cdots \overline{F}_{I(K\cdots K)}^l) \geqq P(F_1^K)P(F_{2(K)}^K)\cdots P(\overline{F}_{I(K\cdots K)}^l) = P(\overline{F}_1^K)P(\overline{F}_{2(K)}^K)\cdots P(\overline{F}_{I(K\cdots K)}^l)$, a lower bound of $\overline{M}$ can be derived as

$$\begin{aligned}\overline{M} &\geqq 1 + \sum_{i=1}^{K}(a_i - a_{i-1})P(\overline{F}_1^i) \\ &\quad + \sum_{j=1}^{K}(a_j - a_{j-1})P(F_1^K \overline{F}_{2(K)}^j) \\ &\quad + \sum_{k=1}^{K}(a_k - a_{k-1})P(F_1^K F_{2(K)}^K \overline{F}_{3(KK)}^k) + \cdots \\ &\geqq 1 + \sum_{i=1}^{K}(a_i - a_{i-1})P(\overline{F}_1^i) \\ &\quad + P(\overline{F}_1^K)\sum_{j=1}^{K}(a_j - a_{j-1})P(\overline{F}_{2(K)}^j) \\ &\quad + P(\overline{F}_1^K)P(\overline{F}_{2(K)}^K) \\ &\quad \cdot\sum_{k=1}^{K}(a_k - a_{k-1})P(\overline{F}_{3(KK)}^k) + \cdots .\end{aligned} \tag{23}$$

On the other hand, by applying inequality

$$\begin{aligned}&P(F_1^i F_{2(i)}^j \cdots \overline{F}_{I(ij\cdots k)}^l) \\ &\quad \leqq \min\{P(F_1^i), P(F_{2(i)}^j), \cdots, P(\overline{F}_{I(ij\cdots k)}^l)\}\end{aligned} \tag{24}$$

to Eq. (22) and noting $\sum_{k=1}^{K}(a_k - a_{k-1}) = a_K = 1$, we obtain the following upper bound of $\overline{M}$.

$$\begin{aligned}\overline{M} &\leqq 1 + \sum_{i=1}^{K}(a_i - a_{i-1})P(\overline{F}_1^i) \\ &\quad + \sum_{i=1}^{K}\min\{P(F_1^i), \sum_{j=1}^{K}(a_j - a_{j-1})P(\overline{F}_{2(i)}^j)\} \\ &\quad + \sum_{i=1}^{K}\sum_{j=1}^{K}\min\{P(F_1^i), P(F_{2(i)}^j),\end{aligned}$$

$$\sum_{k=1}^{K}(a_k - a_{k-1})P(\overline{F}^k_{3(ij)})\} + \cdots. \quad (25)$$

The bounds of $P(F^l_{I(ij\cdots k)})$ and $P(\overline{F}^l_{I(ij\cdots k)})$ will be evaluated in Sect. 4.3.

4.2 Average Bit Error Rate

Let $E_{I(ij\cdots k)}$ be the event that errors occur when $\tilde{Y}^{I(ij\cdots k)}$ is decoded by the ordinary Viterbi decoder. Then, the event error probability P_E is bounded above as follows.

$$\begin{aligned}
P_E &= P(S_1E_1) + \sum_{i=1}^{K} P(F^i_1 S_{2(i)} E_{2(i)}) \\
&\quad + \sum_{i=1}^{K}\sum_{j=1}^{K} P(F^i_1 F^j_{2(i)} S_{3(ij)} E_{3(ij)}) + \cdots \\
&= \{P(E_1) - \sum_{i=1}^{K} P(F^i_1 E_1)\} \\
&\quad + \sum_{i=1}^{K}\{P(F^i_1 E_{2(i)}) - \sum_{j=1}^{K} P(F^i_1 F^j_{2(i)} E_{2(i)})\} \\
&\quad + \sum_{i=1}^{K}\sum_{j=1}^{K}\{P(F^i_1 F^j_{2(i)} E_{3(ij)}) \\
&\quad - \sum_{k=1}^{K} P(F^i_1 F^j_{2(i)} F^k_{3(ij)} E_{3(ij)})\} + \cdots \\
&= P(E_1) - \sum_{i=1}^{K} P(F^i_1)\{P(E_1|F^i_1) - P(E_{2(i)}|F^i_1)\} \\
&\quad - \sum_{i=1}^{K}\sum_{j=1}^{K} P(F^i_1 F^j_{2(i)}) \\
&\quad \cdot\{P(E_{2(i)}|F^i_1 F^j_{2(i)}) - P(E_{3(ij)}|F^i_1 F^j_{2(i)})\} + \cdots \\
&< P(E_1) - \sum_{i=1}^{K} P(F^i_1)\{P(E_1|F^i_1) - P(E_{2(i)}|F^i_1)\} \\
&\quad - \sum_{j=1}^{K} P(F^K_1 F^j_{2(K)})\{P(E_{2(K)}|F^K_1 F^j_{2(K)}) \\
&\quad - P(E_{3(Kj)}|F^K_1 F^j_{2(K)})\} + \cdots
\end{aligned}$$

where the inequality holds because the packet combining improves SNR and, therefore, each braces term is positive. By using inequalities $P(E_{I(K\cdots K)}|F^K_1 \cdots F^l_{I(K\cdots K)}) \geqq P(E_{I(K\cdots K)})$, $P(F^K_1 \cdots F^l_{I-1(K\cdots K)} E_{I(K\cdots Kl)}) \leqq P(E_{I(K\cdots Kl)})$, $P(F^K_1 F^K_{2(K)} \cdots F^l_{I(K\cdots K)}) \geqq P(F^K_1)P(F^K_{2(K)}) \cdots P(F^l_{I(K\cdots K)})$, we obtain

$$\begin{aligned}
P_E &< P(E_1) - \sum_{i=1}^{K-1} P(F^i_1)\{P(E_1|F^i_1) - P(E_{2(i)}|F^i_1)\} \\
&\quad - P(F^K_1)\{P(E_1|F^K_1) - P(E_{2(K)}|F^K_1)\} \\
&\quad - \sum_{j=1}^{K-1} P(F^K_1)P(F^j_{2(K)}) \\
&\quad \cdot\{P(E_{2(K)}|F^K_1 F^j_{2(K)}) - P(E_{3(Kj)}|F^K_1 F^j_{2(K)})\} \\
&\quad - P(F^K_1 F^K_{2(K)})\{P(E_{2(K)}|F^K_1 F^K_{2(K)}) \\
&\quad - P(E_{3(KK)}|F^K_1 F^K_{2(K)})\} - \cdots \\
&< P(E_1) - \sum_{i=1}^{K-1} P(F^i_1)\{P(E_1) - P(E_{2(i)}|F^i_1)\} \\
&\quad - P(F^K_1)P(E_1) + P(E_{2(K)}) \\
&\quad - \sum_{j=1}^{K-1} P(F^K_1)P(F^j_{2(K)}) \\
&\quad \cdot\{P(E_{2(K)}) - P(E_{3(Kj)}|F^K_1 F^j_{2(K)})\} \\
&\quad - P(F^K_1)P(F^K_{2(K)})P(E_{2(K)}) \\
&\quad + P(E_{3(KK)}) - \cdots.
\end{aligned}$$

Furthermore, $P(E_{I(K\cdots Kl)}|F^K_1 \cdots F^K_{I-2} F^l_{I-1}) \leqq P(E_{I(K\cdots Kl)}|\overline{\eta}_{I-1} = \mathrm{T}_l)$, $l \leqq K-1$, holds because $\overline{\eta}_{I-1} = \mathrm{T}_i$ is the worst condition in the case of $\overline{\eta}_I \in R_l$. Hence we have

$$\begin{aligned}
P_E &< P(E_1)\{1 - P(\overline{F}^1_1)\} \\
&\quad + \sum_{i=1}^{K-1} P(F^i_1)P(E_{2(i)}|\overline{\eta}_1 = \mathrm{T}_i) \\
&\quad + P(E_{2(K)})\{1 - P(\overline{F}^K_1)P(\overline{F}^1_{2(K)})\} \\
&\quad + \sum_{j=1}^{K-1} P(\overline{F}^K_1)P(F^j_{2(K)})P(E_{3(Kj)}|\overline{\eta}_2 = \mathrm{T}_j) \\
&\quad + P(E_{3(KK)}) \\
&\quad \cdot\{1 - P(\overline{F}^K_1)P(\overline{F}^K_{2(K)})P(\overline{F}^1_{3(KK)})\} \\
&\quad + \sum_{k=1}^{K-1} P(\overline{F}^K_1)P(\overline{F}^K_{2(K)})P(F^k_{3(KK)}) \\
&\quad \cdot P(E_{4(KKk)}|\overline{\eta}_3 = \mathrm{T}_k) + \cdots. \quad (26)
\end{aligned}$$

In the similar way, the bit error rate P_B is bounded by

$$\begin{aligned}
P_B &< P_{B_1}\{1 - P(\overline{F}^1_1)\} + \sum_{i=1}^{K-1} P(F^i_1)P_{B_{2(i)}|\overline{\eta}_1=\mathrm{T}_i} \\
&\quad + P_{B_{2(K)}}\{1 - P(\overline{F}^K_1)P(\overline{F}^1_{2(K)})\} \\
&\quad + \sum_{j=1}^{K-1} P(\overline{F}^K_1)P(F^j_{2(K)})P_{B_{3(Kj)}|\overline{\eta}_2=\mathrm{T}_j} \\
&\quad + P_{B_{3(KK)}}\{1 - P(\overline{F}^K_1)P(\overline{F}^K_{2(K)})P(\overline{F}^1_{3(KK)})\} \\
&\quad + \sum_{k=1}^{K-1} P(\overline{F}^K_1)P(\overline{F}^K_{2(K)})P(F^k_{3(KK)}) \\
&\quad \cdot P_{B_{4(KKk)}|\overline{\eta}_3=\mathrm{T}_k} + \cdots,
\end{aligned}$$

where $P_{B_{I(ij\cdots k)}}$ is the bit error rate of $\tilde{Y}^{I(ij\cdots k)}$ for the ordinary Viterbi decoding and $P_{B_{I(KK\cdots Kk)}|\overline{\eta}_{I-1}=\mathrm{T}_k}$ is the bit error rate under the condition $\overline{\eta}_{I-1} = \mathrm{T}_k$. The bounds of $P(E_{I(ij\cdots k)})$, $P(E_{I(ij\cdots k)}|\overline{\eta}_{I-1} = \mathrm{T}_k)$, $P_{B_{I(ij\cdots k)}}$, and $P_{B_{I(ij\cdots k)}|\overline{\eta}_{I-1}=\mathrm{T}_k}$ will be evaluated in Sect. 4.4.

4.3 Retransmission Probability

In the PR scheme, each bit of X may not be retransmitted same times. However, the difference is at most one because the retransmitted positions are shifted cyclically. Assume that the two parts of X, say X_1 with $N-M$ bits and X_2 with M bits, are transmitted s and $s+1$ times, respectively. Let random variables $\tilde{Y}_l^I$, $\tilde{X}_l^I$ and $\tilde{y}_{lj}^I = x_j + \tilde{n}_{lj}^I$ correspond to X_l, $l = 1, 2$. Then $\tilde{n}_{1j}^I$ and $\tilde{n}_{2j}^I$ become the Gaussian random variables with mean zero and variance $\mathcal{N}_0/2s$ and $\mathcal{N}_0/2(s+1)$, respectively. Furthermore, letting $d_1 = d(X_1, \tilde{X}_1^I)$ and $d_2 = d(X_2, \tilde{X}_2^I)$, the estimated SNR becomes

$$\overline{\eta}_I = \frac{\mathcal{E}N}{\begin{array}{l}2\sum_{c=1}^{N-M-d_1}\left(\tilde{n}_{1j_c}^I\right)^2 + 2\sum_{e=1}^{d_1}(\tilde{n}_{1j_e}^I - 2\sqrt{\mathcal{E}})^2 + \\ 2\sum_{c=1}^{M-d_2}\left(\tilde{n}_{2j_c}^I\right)^2 + 2\sum_{e=1}^{d_2}(\tilde{n}_{2j_e}^I - 2\sqrt{\mathcal{E}})^2\end{array}}. \quad (27)$$

By applying Lemma A2 in Appendix A to Eq. (27), the following inequality holds

$$\begin{aligned} &P(\overline{\eta}_I < \mathrm{T}_{l-1}|\tilde{X}^I \neq X) \\ &\quad > P(\overline{\eta}_I < \mathrm{T}_{l-1}|\tilde{X}^I = X) \\ &\quad \approx P\left(\sum_{j=1}^{N-M}\left(\tilde{n}_{1j}^I\right)^2 + \sum_{j=1}^{M}\left(\tilde{n}_{2j}^I\right)^2 > \frac{N\mathcal{E}}{2\mathrm{T}_{l-1}}\right). \quad (28)\end{aligned}$$

Furthermore, when $N-M$ and M are large, the distribution of $\sum_{j=1}^{N-M}\left(\tilde{n}_{1j}^I\right)^2$ and $\sum_{j=1}^{M}\left(\tilde{n}_{2j}^I\right)^2$ can well be approximated by the Gaussian distribution. Hence, $P(\overline{F}_{I(ij\cdots k)}^l)$ can be bounded by

$$\begin{aligned} P(\overline{F}_{I(ij\cdots k)}^l) &= \sum_{\tilde{X}^I} P(\tilde{X}^I)P(\overline{\eta}_I < \mathrm{T}_{l-1}|\tilde{X}^I) \\ &> P(\overline{\eta}_I < \mathrm{T}_{l-1}|\tilde{X}^I = X) \\ &\approx Q\left[\frac{\frac{N}{\mathrm{T}_{l-1}}\eta - \{\frac{N-M}{s} + \frac{M}{s+1}\}}{\sqrt{2\{\frac{N-M}{s^2} + \frac{M}{(s+1)^2}\}}}\right]. \quad (29)\end{aligned}$$

In the similar way as Eq. (17), $P(\overline{F}_{I(ij\cdots k)}^l)$ and $P(F_{I(ij\cdots k)}^l)$ are bounded as follows.

$$\begin{aligned} &P(\overline{F}_{I(ij\cdots k)}^l) \\ &\leqq P(E_{I(ij\cdots k)}) \\ &\quad + (1 - P(E_{I(ij\cdots k)}))P(\overline{\eta}_I < \mathrm{T}_{l-1}|\tilde{X}^I = X) \\ &\approx P(E_{I(ij\cdots k)})\left[1 - Q\left[\frac{\frac{N}{\mathrm{T}_{l-1}}\eta - \{\frac{N-M}{s} + \frac{M}{s+1}\}}{\sqrt{2\{\frac{N-M}{s^2} + \frac{M}{(s+1)^2}\}}}\right]\right] \\ &\quad + Q\left[\frac{\frac{N}{\mathrm{T}_{l-1}}\eta - \{\frac{N-M}{s} + \frac{M}{s+1}\}}{\sqrt{2\{\frac{N-M}{s^2} + \frac{M}{(s+1)^2}\}}}\right], \quad (30)\end{aligned}$$

$$\begin{aligned} &P(F_{I(ij\cdots k)}^l) \\ &= \sum_{\tilde{X}^I} P(\tilde{X}_I)P(\mathrm{T}_l \leqq \overline{\eta}_I < \mathrm{T}_{l-1}|\tilde{X}^I) \\ &\leqq \sum_{\tilde{X}^I \neq X} P(\tilde{X}^I)P(\mathrm{T}_l \leqq \overline{\eta}_I < \mathrm{T}_{l-1}|\tilde{X}^I) \\ &\quad + \left(1 - \sum_{\tilde{X}^I \neq X} P(\tilde{X}^I)\right) \\ &\quad \cdot P(\mathrm{T}_l \leqq \overline{\eta}_I < \mathrm{T}_{l-1}|\tilde{X}^I = X) \\ &\quad + \sum_{\tilde{X}^I \neq X} P(\tilde{X}^I) \\ &\quad \cdot \left\{P(\overline{\eta}_I < \mathrm{T}_l|\tilde{X}^I) - P(\overline{\eta}_I < \mathrm{T}_l|\tilde{X}^I = X)\right\} \\ &\leqq P(E_{I(ij\cdots k)})\{1 - P(\overline{\eta}_I < \mathrm{T}_{l-1}|\tilde{X}^I = X)\} \\ &\quad + P(\mathrm{T}_l \leqq \overline{\eta}_I < \mathrm{T}_{l-1}|\tilde{X}^I = X) \\ &\approx P(E_{I(ij\cdots k)})\left[1 - Q\left[\frac{\frac{N}{\mathrm{T}_{l-1}}\eta - \{\frac{N-M}{s} + \frac{M}{s+1}\}}{\sqrt{2\{\frac{N-M}{s^2} + \frac{M}{(s+1)^2}\}}}\right]\right] \\ &\quad + Q\left[\frac{\frac{N}{\mathrm{T}_{l-1}}\eta - \{\frac{N-M}{s} + \frac{M}{s+1}\}}{\sqrt{2\{\frac{N-M}{s^2} + \frac{M}{(s+1)^2}\}}}\right] \\ &\quad - Q\left[\frac{\frac{N}{\mathrm{T}_l}\eta - \{\frac{N-M}{s} + \frac{M}{s+1}\}}{\sqrt{2\{\frac{N-M}{s^2} + \frac{M}{(s+1)^2}\}}}\right], \quad (31)\end{aligned}$$

where the first inequality follows from Eq. (28) and the second inequality holds because $P(E_{I(ij\cdots k)}) = \sum_{\tilde{X}^I \neq X} P(\tilde{X}^I)$ and $P(\overline{\eta}_I < \mathrm{T}_{l-1}|\tilde{X}^I) \leqq 1$.

4.4 Upper Bounds of $P_{B_{I(ij\cdots k)}}$

Since each bit of X is transmitted s or $s+1$ times, the pairwise error probability of an incorrect path $\hat{X}_L$ depends on the position where $\hat{X}_L$ diverges from X. For a convolutional code with rate n/m, each branch of the trellis has m bits. Hence, such dependency has cycle $W = K \cdot m/(\gcd(K, m))^2$ where $\gcd(K, m)$ is the greatest common divisor of K and m, and the bit error rate is bounded by

$$\begin{aligned} P_{B_{I(ij\cdots k)}} \leqq \frac{1}{W}\sum_{w=1}^{W}\sum_{L=1}^{\infty}\sum_{\hat{X}_{Lw}} A(X, \hat{X}_{Lw}) \\ \cdot P(m(\tilde{Y}^I, X) - m(\tilde{Y}^I, \hat{X}_{Lw}) < 0)\end{aligned}$$

$$= \frac{1}{W}\sum_{w=1}^{W}\sum_{L=1}^{\infty}\sum_{\hat{X}_{Lw}} A(X,\hat{X}_{Lw}) \cdot P\left(\sum_{l=1}^{L}(x_{j_l}-\tilde{x}_{j_l})\tilde{y}_{j_l}^{I}<0\right), \quad (32)$$

where $\hat{X}_{Lw}$ denotes an incorrect path that diverges at the w-th position of the cycle. Let $d = d(X,\hat{X}_{Lw})$, i.e. d bits of $\hat{X}_{Lw}$ differ from X. Assume that d_1 and d_2 bits of such d bits are transmitted s and $s+1$ times, respectively. In this case, $\sum_{j=1}^{N}(x_j-\hat{x}_j)\tilde{y}_j^I$ becomes the Gaussian random variable with mean $-2\mathcal{E}d$ and variance $4\mathcal{E}\frac{N_0}{2}\left(\frac{d_1}{s}+\frac{d_2}{s+1}\right)$. Hence, Eq. (32) becomes

$$\begin{aligned} P_{B_{I(ij\cdots k)}} &< \frac{1}{W}\sum_{w=1}^{W}\sum_{L=1}^{\infty}\sum_{\hat{X}_{Lw}} A(X,\hat{X}_{Lw}) \\ &\cdot \int_{-\infty}^{0}\frac{1}{\sqrt{2\pi 4\mathcal{E}\frac{N_0}{2}(\frac{d_1}{s}+\frac{d_2}{s+1})}} \\ &\cdot \exp\left\{-\frac{(t-2\mathcal{E}d)^2}{2\cdot 4\mathcal{E}\frac{N_0}{2}(\frac{d_1}{s}+\frac{d_2}{s+1})}\right\}dt \\ &= \frac{1}{W}\sum_{w=1}^{W}\sum_{L=1}^{\infty}\sum_{\hat{X}_{Lw}} A(X,\hat{X}_{Lw})Q\left\{\sqrt{2\eta\frac{d^2}{\frac{d_1}{s}+\frac{d_2}{s+1}}}\right\}. \end{aligned} \quad (33)$$

We also obtain

$$\begin{aligned} P(E_{I(ij\cdots k)}) &\leqq 1-(1-P_{E_{I(ij\cdots k)}})^{N-\kappa} \\ &\leqq 1-\left(1-\frac{1}{W}\sum_{w=1}^{W}\sum_{L=1}^{\infty}\sum_{\hat{X}_{Lw}} Q\left\{\sqrt{2\eta\frac{d^2}{\frac{d_1}{s}+\frac{d_2}{s+1}}}\right\}\right)^{N-\kappa}, \end{aligned} \quad (34)$$

where $P_{E_{I(ij\cdots k)}}$ denotes the event error probability of $\tilde{Y}^{I(ij\cdots k)}$ for the ordinary Viterbi decoding.

Under the condition $\bar{\eta}_{I-1} = \mathrm{T}_l$, $\sum_{j=1}^{N}(x_j-\hat{x}_j)\tilde{y}_j^I$ becomes the Gaussian random variable with mean $-2\mathcal{E}d$ and variance $4\mathcal{E}\left\{\frac{\mathcal{E}}{2\mathrm{T}_l}d_1+\left(\frac{\mathcal{E}}{2\mathrm{T}_l}s^2+\frac{N_0}{2}\right)\frac{d_2}{(s+1)^2}\right\}$. Hence, $P_{B_{I(ij\cdots k)}|\bar{\eta}_{I-1}=\mathrm{T}_k}$ and $P(E_{I(ij\cdots k)}|\bar{\eta}_{I-1}=\mathrm{T}_k)$ can bounded above as follows.

$$\begin{aligned} P_{B_{I(ij\cdots k)}|\bar{\eta}_{I-1}=\mathrm{T}_k} &\leqq \frac{1}{W}\sum_{w=1}^{W}\sum_{L=1}^{\infty}\sum_{\hat{X}_{Lw}} A(X,\hat{X}_{Lw}) \\ &\cdot Q\left\{\sqrt{\frac{2d^2\mathrm{T}_k(s+1)^2}{(s+1)^2d_1+\left(s^2+\frac{\mathrm{T}_k}{\eta}\right)d_2}}\right\}, \end{aligned} \quad (35)$$

$$\begin{aligned} P(E_{I(ij\cdots k)}|\bar{\eta}_{I-1}=\mathrm{T}_k) &\leqq 1-\left(1-\frac{1}{W}\sum_{w=1}^{W}\sum_{L=1}^{\infty}\sum_{\hat{X}_{Lw}} Q\left\{\sqrt{\frac{2d^2\mathrm{T}_k(s+1)^2}{(s+1)^2d_1+\left(s^2+\frac{\mathrm{T}_k}{\eta}\right)d_2}}\right\}\right)^{N-\kappa}. \end{aligned} \quad (36)$$

5. Numerical Results

In Figs. 3 and 4, the theoretical bounds of bit error rate and throughput are depicted with simulation results. We used the convolutional code with generator polynomial $G = (1+D+D^2, 1+D^2)$. Hence, the constraint length is three and the rate is 1/2. The packet length N is assumed to be 1000 bits and the threshold T is set to 2.5 dB which corresponds to $P_B \approx 3.0\times10^{-5}$. In the PR scheme, we used $K=4$ as shown in Fig. 1. Hence, if $\bar{\eta}_I$ is in R_l, $l=1,2,3,4$, then $N_l = N\times(l/4) = 250\times l$ bits are retransmitted. The threshold of each retransmission region R_l, i.e. T_l, is shown in Table 1, which is determined based on the rule given in Appendix B.

Figure 3 shows that both WR and PR schemes can attain the given bit error rate tolerance for any SNR. But, in the WR scheme, the bit error rate becomes much lower than the tolerance in some regions of SNR, and the throughput is terraced in such regions as show in Fig. 4. This means that the WR scheme retransmits too

Table 1 Retransmission threshold.

I	T_1^I(dB)	T_2^I(dB)	T_3^I(dB)
1	2.169	1.505	0.715
2	2.367	1.955	1.497
3	2.463	2.165	1.842
4	-	2.286	2.037
5	-	2.365	2.163
6	-	2.421	2.250

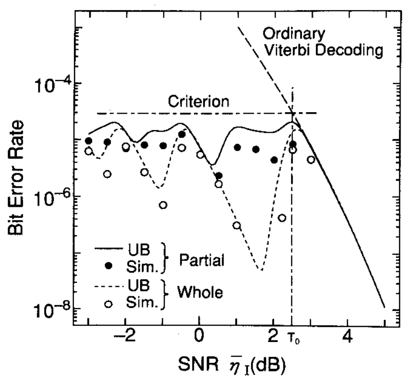

Fig. 3 Bit error rate.

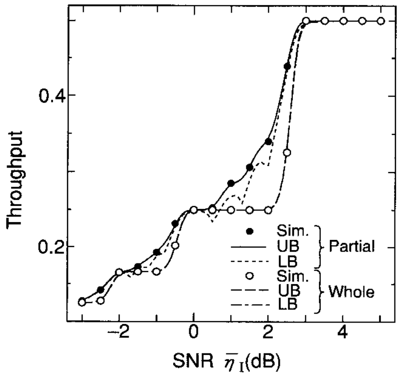

Fig. 4 Throughput.

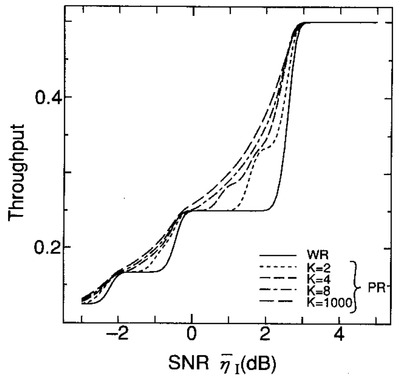

Fig. 5 Throughputs of PR scheme with $K = 2, 4, 8, 1000$.

many bits in such regions. On the other hand, in the PR scheme, the throughput decreases smoothly as the SNR becomes worse and the throughput is considerably improved compared with the WR scheme.

We note that concerning the throughput of the PR scheme shown in Fig. 4, the theoretical upper bound is tight, but the theoretical lower bound waves. This is caused by a little loose bound of Eq. (24). But it seems to be difficult to derive a tighter bound which can be easily calculated.

The throughput of the PR scheme is improved as the number of subblocks K increases because the length of retransmission packet can be precisely controlled based on the estimated SNR. However, the larger K becomes, the more bits must be transmitted in the feedback channel, which means that the error control for the feedback channel becomes sever. In Fig. 5, the theoretical upper bounds of throughput are shown for various K, where $K = N = 1000$ is the best case. From the figure, $K = 4$ or 8 almost attain the best throughput.

In the PR scheme, the load of the Viterbi decoder is another important factor to be considered. See Appendix B. The expected number of the Viterbi decoding necessary to accept X is given by substituting 1 into a_l in Eq. (20) while it is given by Eq. (4) for the WR

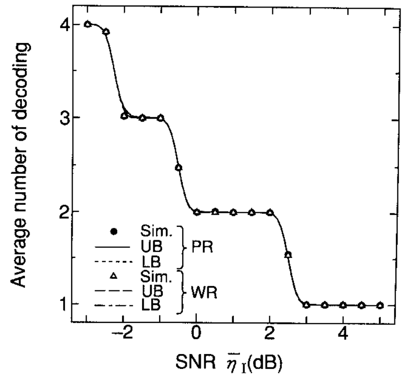

Fig. 6 Average number of Viterbi decoding.

scheme. By the similar bounds as Eqs. (5), (6), (23), and (25), the expected numbers of the Viterbi decoding are calculated and depicted in Fig. 6. This figure shows that the PR scheme can improve the throughput with almost no increasing the load of the Viterbi decoder.

6. Conclusion

We proposed a new hybrid-ARQ scheme with retransmission criterion based on an estimated decoding error rate, and we evaluated the performance by the theoretical bounds and computer simulations. From the evaluation, we showed that the WR and PR schemes can attain a given bit error rate tolerance with relatively high throughput for any SNR, and the PR scheme can improve the throughput compared with the WR scheme without increasing the load of the Viterbi decoder. Furthermore, the derived theoretical bounds were also shown to be tight.

In this paper, we assumed that the channel is slow time-varying and the BPSK is used. However, we note that the proposed ARQ scheme and the technique for the analyses of the performance can be applied to trellis coded modulation (TCM) schemes especially if the TCM schemes are uniform. Furthermore, they can be applied to moderately time-variant channels by modifying the scheme slightly [14].

We point out that the similar hybrid-ARQ scheme can be constructed by using a block code instead of a convolutional code. However, the SNR of the channel can easily and precisely be estimated for a wide range by the convolutional code than the block codes. Hence, the convolutional code seems to be suited for this kind of hybrid-ARQ scheme.

We estimate the SNR of the channel from the combined packet $\tilde{Y}^I$ and the decoded packet $\tilde{X}^I$ as shown in Eq. (2). However, the SNR (or bit error rate) may be estimated from other parameters, e.g. the ratio of likelihood between the best two paths, the ratio between the likelihood of the best path and the sum of the others, etc. If such estimation is possible and valid for a wide range of the SNR, it can be used instead of our estimator and, hence, our shceme should be compared with such schemes in future studies. But, we believe that our simple estimator of the SNR is not inferior to other complicated estimator.

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Appendix A

Lemma A1

Let $U \triangleq \sum_{j=1}^{N} \left(\tilde{n}_j^I\right)^2$ and $V \triangleq \sum_{c=1}^{N-d(X,\tilde{X}^I)} \left(\tilde{n}_{j_c}^I\right)^2 + \sum_{e=1}^{d(X,\tilde{X}^I)} \left(\tilde{n}_{j_e}^I - 2\sqrt{\mathcal{E}}\right)^2$. Then, for any $\beta > 0$,

$$P(U > \beta) < P(V > \beta) \qquad \text{(A·1)}$$

Proof: The probability density function of U and V are given as follows[15].

$$p_U(t) = \frac{1}{\left(\frac{N_0}{2I}\right)^{\frac{N}{2}} 2^{\frac{N}{2}} \Gamma\left(\frac{N}{2}\right)} t^{\frac{N}{2}-1} \exp\left(-\frac{t}{2\frac{N_0}{2I}}\right) \qquad \text{(A·2)}$$

$$p_V(t) = \frac{1}{\left(\frac{N_0}{2I}\right)^{\frac{N}{2}} 2^{\frac{N}{2}}} t^{\frac{N}{2}-1} \exp\left(-\frac{t+s^2}{2\frac{N_0}{2I}}\right) \cdot \sum_{k=0}^{\infty} \frac{t^k \left(\frac{s}{2\frac{N_0}{2I}}\right)^{2k}}{k!\Gamma\left(\frac{N}{2}+k\right)}, \qquad \text{(A·3)}$$

where $s^2 = 4\mathcal{E}d(X, \tilde{X}^I)$. From Eqs. (A·2) and (A·3), we note that $p_U(0) = p_V(0) = 0$. Furthermore, we can easily show that there exist $\alpha > 0$ such that

$$\begin{cases} p_U(t) > p_V(t), & 0 < t < \alpha \\ p_U(t) = p_V(t), & t = \alpha \\ p_U(t) < p_V(t), & t > \alpha. \end{cases}$$

From the above inequalities and $\int_\beta^\infty p(t)dt = 1 - \int_0^\beta p(t)dt$, we have

$$\int_\beta^\infty p_U(t)dt < \int_\beta^\infty p_V(t)dt \qquad \text{(A·4)}$$

for any $\beta > 0$.

Lemma A2

Let $U_1 \triangleq \sum_{j=1}^{N-M} (\tilde{n}_{1j}^I)^2$, $U_2 \triangleq \sum_{j=1}^{M} (\tilde{n}_{2j}^I)^2$, $V_1 \triangleq \sum_{c=1}^{N-M-d_1} (\tilde{n}_{1j_c}^I)^2 + \sum_{e=1}^{d_1} (\tilde{n}_{1j_e}^I - 2\sqrt{\mathcal{E}})^2$, and $V_2 \triangleq \sum_{c=1}^{M-d_2} (\tilde{n}_{2j_c}^I)^2 + \sum_{e=1}^{d_2} (\tilde{n}_{2j_e}^I - 2\sqrt{\mathcal{E}})^2$. Then

$$P(U_1 + U_2 > \beta) < P(V_1 + V_2 > \beta). \qquad \text{(A·5)}$$

Proof: Letting $[t]_+ = \max[t, 0]$, we have

$$P(V_1 + V_2 > \beta) = \int_0^\infty p_{V_1}(a)da \int_{[\beta-a]_+}^\infty p_{V_2}(b)db$$

$$\begin{aligned} &> \int_0^\infty p_{V_1}(a)da \int_{[\beta-a]_+}^\infty p_{U_2}(b)db \\ &= P(V_1+U_2>\beta) \\ &= \int_0^\infty p_{U_2}(a)da \int_{[\beta-a]_+}^\infty p_{V_1}(b)db \\ &> \int_0^\infty p_{U_2}(a)da \int_{[\beta-a]_+}^\infty p_{U_1}(b)db \\ &= P(U_1+U_2>\beta), \qquad (A\cdot 6) \end{aligned}$$

where the inequalities follows from Eq. (A· 4).

Appendix B

If the thresholds $\{T_l\}$ are not set properly in the PR scheme, it may happen that partial packets are retransmitted many times. Although such many retransmissions of partial packets do not make worse the throughput because the selective ARQ scheme is considered, the load of the Viterbi decoder becomes heavy. In such case, it is obviously preferable that a partial packet is retransmitted only once after several retransmissions of the whole packet. Since the SNR of $\tilde{Y}^I$ increases as I does, the threshold T_l should be changed as I. Let T_l^I be the threshold used for $\tilde{Y}^I$. In order to determine T_l^I, we consider $\tilde{Y}^{I+1(K\cdots Kl)}$ which is constructed from I whole packets $q_1 q_2^K \cdots q_I^K$ and one partial packet q_{I+1}^l. The acceptance probability of $\tilde{Y}^{I+1(K\cdots Kl)}$, i.e. $P(S_{I+1(K\cdots Kl)})$, is given by

$$\begin{aligned} &P(S_{I+1(K\cdots Kl)}) \\ &= 1 - P(\overline{F}^1_{I+1(K\cdots K)l}) \\ &\lesssim 1 - Q\left[\frac{\frac{N}{T_0}\eta - \{\frac{N-N_l}{I} + \frac{N_l}{I+1}\}}{\sqrt{2\{\frac{N-N_l}{I^2} + \frac{N_l}{(I+1)^2}\}}}\right], \qquad (A\cdot 7) \end{aligned}$$

where the last inequality follows from Eq. (29) with $s = I$ and $M = N_l$. If $P(S_{I+1(K\cdots Kl)})$ is almost one, say r, then partial retransmission is almost not repeated. Let η_r^I be the SNR such that the right side of Eq. (A· 7) is equal to r. Then, the threshold T_l^I for $\tilde{Y}^I$ can be set as $T_l^I = I\eta_r^I$. We used $r = 0.9$ in Figs. 3 and 4. If $T_l^I > T_0$, the corresponding retransmission region R_l vanishes. The values of T_l^I are shown in Table 1.

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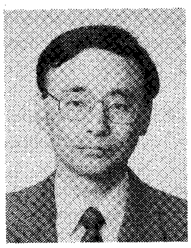

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